On a Special Elliptic Ruled Surface of the Ninth Order.

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The object of this paper is to discuss the following problem connected with a tetrahedron:

Are there lines connected with a tetrahedron, such that if the vertices are reflected in these lines, the reflections will fall on the opposite faces? If so, what is the locus of these lines?

It has been shown by G. T. Bennett* that for any given tetrahedron there are ∞^1 such lines, and that when the opposite edges are equal, the locus consists of three cylindroids. The existence of these lines will be established by direct geometric considerations, and their locus will be discussed together with other related questions.

§ 1. *Special Displacements of a Given Tetrahedron.*

Let $(\alpha x)(a\eta)=0$ be a collineation between a point x and a point x'. To within a factor of proportionality, the coefficients of the η's are the coordinates of the point x', *i. e.*, $kx'_i=a_i(\alpha x)$, where $i=0, 1, 2, 3$, and where k is the factor of proportionality. If we put $x'_i=x_i$, the eliminant of the four equations, giving the fixed point, is

$$\begin{vmatrix} a_0\alpha_0-k & a_0\alpha_1 & a_0\alpha_2 & a_0\alpha_3 \\ a_1\alpha_0 & a_1\alpha_1-k & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_0 & a_2\alpha_1 & a_2\alpha_2-k & a_2\alpha_3 \\ a_3\alpha_0 & a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3-k \end{vmatrix}=0,$$

which may be written as

$$k^4-I_1k^3+I_2k^2-I_3k+I_4=0,$$

where

$$I_1=(a\alpha)=a_0\alpha_0+a_1\alpha_1+a_2\alpha_2+a_3\alpha_3,\qquad I_2=(\alpha\beta\cdot ab),$$
$$I_3=(\alpha\beta\gamma\cdot abc),\qquad I_4=(\alpha\beta\gamma\delta\cdot abcd).$$

We are concerned with those collineations for which $(\alpha a)=0$.†

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† These are Study's "Normal Collineations."

Suppose the above collineation to be a displacement. The normal form of a displacement is

$$x' = e^{ia}x,$$
$$\bar{x}' = e^{-ia}\bar{x},$$
$$z' = z + p\alpha,$$

where p is the pitch of the screw, α the angle turned through, and the barred variables are the conjugates of the unbarred variables.

Written homogeneously these equations become

$$x'_0 = e^{ia}x_0,$$
$$x'_1 = e^{-ia}x_1,$$
$$x'_2 = x_2 + p\alpha x_3,$$
$$x'_3 = x_3,$$

for which I_1 is $e^{ia}+e^{-ia}+2$ or $I_1=2+2\cos\alpha$.

For a displacement I_1 will equal 0 when $\alpha=180°$. Thus we see that

There are displacements of T which send each vertex onto the face opposite it, and these displacements consist of rotations about an axis, through an angle of 180°, *and a translation.*

The general collineation has 15 independent constants. In the displacement herein considered, the plane at infinity (which we shall name W), the absolute in this plane and the size of T are all left unaltered. This puts $3+5+1$ or 9 conditions on the displacement. It is one condition for each vertex to go onto the face opposite it. This leaves $15-13=2$ constants at our disposal and consequently ∞^2 axes of rotation. Taking the pitch of the screw motion to be 0, which is one more condition, leaves ∞^1 axes of rotation and reduces the above displacements to rotations of 180° only, about these ∞^1 axes, and therefore to reflections in these lines.

Throughout this paper T is taken to be the reference tetrahedron, and the one formed by the reflection of the vertices in any one of these ∞^1 lines will be designated as T'. Then for a given T, there are ∞^1 T''s. Or,

For a given tetrahedron T, there exists ∞^1 lines such that reflections of T in these lines give ∞^1 tetrahedrons T', which are inscribed to T.

§ 2. *The Axes of Rotation.*

Consider the four reflections of T, given by

$$\begin{array}{lllll}
R_0: & x'_0=-x_0, & x'_1=x_1, & x'_2=x_2, & x'_3=x_3; \\
R_1: & x'_0=x_0, & x'_1=-x_1, & x'_2=x_2, & x'_3=x_3; \\
R_2: & x'_0=x_0, & x'_1=x_1, & x'_2=-x_2, & x'_3=x_3; \\
R_3: & x'_0=x_0, & x'_1=x_1, & x'_2=x_2, & x'_3=-x_3.
\end{array}$$

R_0 sends a line p, whose coordinates are

$$p_{01},\ p_{02},\ p_{03},\ p_{12},\ p_{23},\ p_{31}$$

into a line whose coordinates are

$$-p_{01},\ -p_{02},\ -p_{03},\ p_{12},\ p_{23},\ p_{31},$$

which we shall call pR_0. R_1 sends p into a line pR_1 whose coordinates are

$$-p_{01},\ p_{02},\ p_{03},\ -p_{12},\ p_{23},\ -p_{31}.$$

If π be an axis with coordinates π_{ij}, meeting pR_0, then

$$-p_{01}\pi_{01}-p_{02}\pi_{02}-p_{03}\pi_{03}+p_{23}\pi_{23}+p_{31}\pi_{31}+p_{12}\pi_{12}=0.$$

If π meets pR_1,

$$-p_{01}\pi_{01}+p_{02}\pi_{02}+p_{03}\pi_{03}+p_{23}\pi_{23}-p_{31}\pi_{31}-p_{12}\pi_{12}=0.$$

If π meets both pR_0 and pR_1, then

$$p_{01}\pi_{01}=p_{23}\pi_{23}, \tag{1}$$

i. e., π belongs to a linear complex. Similarly, if π meets pR_0 and pR_2,

$$p_{02}\pi_{02}=p_{31}\pi_{31}, \tag{2}$$

and if π meets pR_0 and pR_3,

$$p_{03}\pi_{03}=p_{12}\pi_{12}. \tag{3}$$

Or, if π meets pR_0, pR_1, pR_2 and pR_3, then it is common to three linear complexes. Consequently, there are ∞^1 lines π meeting the four lines above, and these four lines pR_i belong to a regulus. p and π play dual rôles and so p must also meet the same four lines.

We now seek the equation of the quadric on which pR_i lie. From (1), (2) and (3) we have for any given line π, equations of type

$$(x_0y_1-x_1y_0)\pi_{01}-(x_2y_3-x_3y_2)\pi_{23}=0.$$

Eliminating y_1, y_2 and y_3,

$$\rho y_0=x_0(x_0^2\pi_{01}\pi_{02}\pi_{03}+x_1^2\pi_{12}\pi_{13}\pi_{10}++)=0,$$

with three similar equations. These equations are satisfied by $x\equiv y$, and by

$$x_0^2\pi_{01}\pi_{02}\pi_{03}+x_1^2\pi_{12}\pi_{13}\pi_{10}+x_2^2\pi_{20}\pi_{21}\pi_{23}+x_3^2\pi_{30}\pi_{31}\pi_{32}=0, \tag{4}$$

which is the desired equation. It represents *a quadric referred to a self-polar tetrahedron of reference*, on which the four lines pR_i must lie, *as well as the lines p and π.*

The relation of p and π in (1), (2) and (3) is that the reflection of a vertex e_i of T in p and π is on ε_i, the face opposite e_i. This reflection can be made a physical one by sending the line π to infinity in a direction perpendicular to p. This makes π the polar line, as to the absolute in W, of the

point of intersection of the line p and the plane W. This makes the quadric (4) touch the plane W, *i. e.*, makes it a paraboloid, and because of the way π is sent to infinity, makes this paraboloid orthic, *i. e.*, makes it apolar to the absolute.

Making a slight change in coordinates we shall write (4) as

$$\left(\frac{x^2}{y}\right) \equiv \frac{x_0^2}{y_0} + \frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} = 0,$$

which is any quadric referred to a self-polar tetrahedron, adding the necessary requirements to make it an orthic paraboloid. To be a paraboloid it must touch the plane at infinity, whose equation is $(x) = 0$. This condition is

$$\frac{1}{y_1 y_2 y_3} + \frac{1}{y_0 y_2 y_3} + \frac{1}{y_0 y_1 y_3} + \frac{1}{y_0 y_1 y_2} = 0, \text{ or } y_0 + y_1 + y_2 + y_3 = 0.$$

To be orthic, *i. e.*, to be apolar to the absolute, the equation of which we shall write as $(A\xi)^2 = 0$, where A_{ii} is the square of the area of the face ε_i of T, the condition is

$$\left(\frac{A^2}{y}\right) \equiv \frac{A_{00}}{y_0} + \frac{A_{11}}{y_1} + \frac{A_{22}}{y_2} + \frac{A_{33}}{y_3} = 0.$$

Then all lines p lie on the quadrics represented by $\left(\frac{x^2}{y}\right) = 0$, where these quadrics are subject to the relations $\left(\frac{A^2}{y}\right) = 0$ and $(y) = 0$, where y is the point of π where the axis of the quadric meets W, *i. e.*, is the point of contact of W and the quadric. The quadric being orthic, its axis and principal generators form three mutually perpendicular lines, which meet W in three points forming a triad, self-conjugate with regard to the absolute. As π is the polar line of the point p (*i. e.*, the point where the line p meets the plane W), and the principal tangent plane meets W in a line which is the polar line of the point y, the line p must be one of the principal generators of the quadric. It is at once seen that both of the principal generators play the same rôle; and, consequently, both are lines p of the kind herein discussed. For either as a line p, the corresponding π is the polar of the meet of p and W as to the absolute, and these two corresponding lines π and π' meet on the point y. So we have this theorem:

All the lines sought are the principal generators of the quadrics $\left(\frac{x^2}{y}\right) = 0$, *where* $(y) = 0$ *and* $\left(\frac{A^2}{y}\right) = 0$.

A case in point is the rectangular hyperboloid $xy=z$. The polar plane of a point x', y', z' is $xy'+x'y=z+z'$. This equation is satisfied by x', $-y'$, $-z'$ and $-x'$, y', $-z'$, the reflections of x', y', z' in the lines $x=0$ and $y=0$, respectively. These two lines are the principal generators of $xy=z$.

§ 3. *The Plane at Infinity and the Surface Sought.*

The reflection of a vertex e_i of T in *any* line at infinity, will lie on the face ε_i, opposite e_i. For, to get a reflection of a point in two lines, one takes the fourth harmonic of this point and the two points where the two given lines are intersected by the line on the given point, meeting the two given lines. As one of the two lines is a line π in the plane W, any other line in W, that is on the meet of π and the intersection of W and ε_i, would do for a line p. For, then the reflection of e_i in p and π would be at their intersection and consequently on ε_i. Therefore, in a certain sense the *entire plane W is* to be considered *a part of the locus of lines* for which we are seeking.*

§ 4. *The Curve at Infinity.*

Let Ω denote the surface on which the lines lie. We seek the intersection of Ω and W. The cubic surface $\left(\dfrac{A^2}{y}\right)$ meets the plane W in a cubic curve, the locus of all points y, where the axes of the quadrics $\left(\dfrac{x^2}{y}\right)=0$ meet the plane W. Also, as before stated, these points y are the intersections of pairs of lines π and π' on which are the points, where the principal generators meet the plane W.

These line pairs π and π' are conjugate as to the absolute and form the degenerate conics of the net of apolar conics of some cubic. The cubic $\left(\dfrac{A^2}{y}\right)=0$, being on the double points of these degenerate conics, is the Jacobian of this net, and the Hessian of the cubic of which this is the apolar net.† The Cayleyan of this latter cubic would be enveloped by these line pairs π and π', and *the reciprocal of this Cayleyan as to the absolute passes through the point-pairs on π and π'.* Therefore:

The intersection of all lines p and the plane W is a cubic curve, which we shall name R. This cubic R meets the absolute in six points, and the tangents to this cubic at these points can be seen at once to be special lines of Ω. The

* This plane W will not be included as a part of Ω in this paper.

† See Clifford's Papers, "Polar Theory of the Plane Cubic."

π's corresponding to these six tangents are the tangents to the absolute, where it meets the cubic R. Moreover, an examination of other possible special lines of Ω in the plane W shows that there are none other than these six. Consequently,

The intersection of the plane W and the surface Ω consists of a cubic curve and six special lines and is therefore of the ninth order.

These six lines meet in fifteen points; they meet the cubic in six points other than the points of tangency; and the six lines meet lines p at the six points of tangency. In all the intersection of the plane W and the surface Ω has 27 double points, one less than the total maximum number for a curve of the ninth order. Therefore,

The intersection of the plane W and the surface Ω is an elliptic curve of the ninth order, and it follows that Ω is an elliptic surface of the ninth order.

§ 5. *Special Lines of Ω.*

There are some lines of Ω that are, in a special way, connected with the tetrahedron T. G. T. Bennett* has mentioned the two following types:

(1) The two perpendicular normals to an edge at its middle point, each equallv inclined to the faces through that edge, *i. e.*, lying in the planes that bisect the dihedral angles on an edge.

(2) Any line which can be drawn, meeting two opposite edges and bisecting, internally or externally, at each of its extremities, the angle subtended by the opposite edge.

It is seen from simple geometric observation that these lines are proper lines of Ω. The six pairs of perpendicular lines of type (1) are evidently pairs of principal generators of six of the quadrics of § 2. For each pair of opposite edges of T there are seven lines of type (2). This is shown by the correspondence set up on the pairs of opposite edges. For, if $\overline{12}$ and $\overline{34}$ are opposite edges of T, then for two points 1 and 2 on one edge, we have one point y on the other edge. This point y definitely determines two points x on the first edge, one on the internal and one on the external bisector of the angle $\underline{|1\,y\,2}$, *i. e.*, for one point y there are four points on the edge $\overline{12}$. Similarly, for one point x on the edge $\overline{12}$ are four points on the edge $\overline{34}$. Therefore, the correspondence between the two edges, thus established is a 4—4 correspondence. The lines $\overline{xy}$ are given by the coincidences of this correspondence which must be eight in number. However, the plane W contains one of these

* See page 431, footnote.

lines, which is not to be included as part of Ω, or there are seven lines of type (2). It is evident that no other lines of Ω meet an edge of T, and consequently,

A given edge of T meets nine lines of the surface Ω—two of type (1) *and seven of type* (2)*—or, what is the same thing, an edge of T meets the surface Ω in nine points, which verifies the fact that Ω is of the ninth order.*

The equation of Ω can be written from this point of view, by expressing the coordinates of any line of Ω in terms of an elliptic parameter. To do this, it is only necessary to suppose the cubic curve R to be expressed in terms of an elliptic parameter u. Then since for each point of R there corresponds one line, in general, the lines can be expressed in terms of this parameter u. The six coordinates p_{ij} of each line of Ω will each equal the product of nine terms like $\sigma(u-a)$, one for each of the lines that meets an edge of the tetrahedron of reference, *i. e.*,

$$p_{ij}=\sigma(u-a_1)\sigma(u-a_2)\ldots\sigma(u-a_9).$$

Since two of the nine lines on an edge of T are perpendicular, their parameters would differ only by a half period, *i. e.*, if one is $(u-\alpha)$, the other would be $(u-\alpha+\omega)$. Moreover, since seven of the nine lines on an edge of T, also meet the opposite edge of T, seven of the σ factors would be the same for the two coordinates p_{ij} and p_{kl}. The coordinates of any line of Ω would then be written:

$$\begin{aligned}
p_{01}&=\sigma(u-a_1)\sigma(u-a_2)\ldots\sigma(u-a_7)\sigma(u-\alpha_{01})\sigma(u-\alpha_{01}+\omega),\\
p_{23}&=\sigma(u-a_1)\sigma(u-a_2)\ldots\sigma(u-a_7)\sigma(u-\alpha_{23})\sigma(u-\alpha_{23}+\omega),\\
p_{02}&=\sigma(u-b_1)\sigma(u-b_2)\ldots\sigma(u-b_7)\sigma(u-\alpha_{02})\sigma(u-\alpha_{02}+\omega),\\
p_{31}&=\sigma(u-b_1)\sigma(u-b_2)\ldots\sigma(u-b_7)\sigma(u-\alpha_{31})\sigma(u-\alpha_{31}+\omega),\\
p_{03}&=\sigma(u-c_1)\sigma(u-c_2)\ldots\sigma(u-c_7)\sigma(u-\alpha_{03})\sigma(u-\alpha_{03}+\omega),\\
p_{12}&=\sigma(u-c_1)\sigma(u-c_2)\ldots\sigma(u-c_7)\sigma(u-\alpha_{12})\sigma(u-\alpha_{12}+\omega).
\end{aligned}$$

§ 6. *Introduction of Elliptic Functions.*

The cubic surface $\left(\frac{A^2}{y}\right)=0$ contains the edges of the reference tetrahedron T. By putting $(y)=0$, and thereby getting the cubic curve which is the intersection of the plane W and this cubic surface, one gets a cubic curve which passes through the six points where the edges of T meet W.

Let this cubic bear an elliptic parameter u, and for the above six points

of this cubic let u have the values a, b, c, x, y and z, respectively. As these six values of the elliptic parameter are collinear by three's, we have

$$a+b+c=0, \quad b+x+y=0,$$
$$a+y+z=0, \quad c+x+z=0.$$

Adding these four equations and substituting 0 for $a+b+c$, the result is $x+y+z=0$, or $=\omega_1$, where ω_1 is a half-period. Then,

$$x=\omega_1-y-z=\omega_1+a, \text{ and similarly, } y=\omega_1+b, \ z=\omega_1+c.$$

Consider $y_0=\lambda_0\sigma(u-a)\sigma(u-b)\sigma(u-c)$. This represents a line on the points whose parameters are a, b and c respectively, where $a+b+c=0$. The line on the points whose parameters are a, y and z respectively, is

$$y_1=\lambda_1\sigma(u-a)\sigma(u-b-\omega_1)\sigma(u-c-\omega_1),$$

and similar equations for the lines that are on b, y, z and c, x, z. The function $\sigma(u-b-\omega_1)$ will be replaced by $\sigma_1(u-b)$, where σ_1 is the *allied sigma-function*. Then we have

$$y_0=\lambda_0\sigma(u-a)\sigma(u-b)\sigma(u-c),$$
$$y_1=\lambda_1\sigma(u-a)\sigma_1(u-b)\sigma_1(u-c), \text{ etc.}$$

It is necessary now to so determine λ_i that $(y)=0$, and $\left(\frac{A^2}{y}\right)=0$. If $(y)=0$,

$$\lambda_0\sigma(u-a)\sigma(u-b)\sigma(u-c)+\lambda_1\sigma(u-a)\sigma_1(u-b)\sigma_1(u-c)$$
$$+\lambda_2\sigma_1(u-a)\sigma(u-b)\sigma_1(u-c)+\lambda_3(\ldots\ldots)=0.$$

Dividing by the coefficient of λ_0,

$$\lambda_0+\lambda_1\frac{\sigma(u-a)\sigma_1(u-b)\sigma_1(u-c)}{\sigma(u-a)\sigma(u-b)\sigma(u-c)}+\lambda_2(\ldots\ldots)+\lambda_3(\ldots\ldots)=0. \tag{5}$$

To remove the infinities for a, b and c, put $u=a$, $u=b$, $u=c$ in succession in (5), observing that $\sigma_1(0)=1$, this gives

$$\lambda_2\frac{\sigma_1(a-c)}{\sigma(a-c)}+\lambda_3\frac{\sigma_1(a-b)}{\sigma(a-b)}=0,$$
$$\lambda_1\frac{\sigma_1(b-c)}{\sigma(b-c)}+\lambda_3\frac{\sigma_1(a-b)}{\sigma(a-b)}=0,$$
$$\lambda_1\frac{\sigma_1(b-c)}{\sigma(b-c)}+\lambda_2\frac{\sigma_1(a-c)}{\sigma(a-c)}=0.$$

Solving for the λ's, we have

$$\lambda_1=\frac{\sigma(b-c)}{\sigma_1(b-c)}, \quad \lambda_2=\frac{\sigma(a-c)}{\sigma_1(a-c)}, \quad \lambda_3=\frac{\sigma(a-b)}{\sigma_1(a-b)},$$

which substituted in (5) reduces that equation to

$$\lambda_0+\frac{\sigma(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)}+(\ldots.)+(\ldots.)=0. \tag{6}$$

To determine λ_0, let $u=a+\omega_1$. Then $u-a=\omega_1$ and $\sigma_1(u-a)=\sigma_1\omega=0$. This eliminates the last two terms in (6), and leaves

$$\lambda_0+\frac{\sigma(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)}=0, \tag{7}$$

whence

$$\lambda_0=(e_1-e_2)\,(e_1-e_3)\,\frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)},$$

which gives as the equation corresponding to $(y)=0$, by substituting in (6),

$$(e_1-e_2)\,(e_1-e_3)\,\frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\sigma_1(c-a)\sigma_1(a-b)} \\ +\sum^{3}\frac{\sigma(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)}=0, \tag{8}$$

or,

$$\left.\begin{aligned} y_0&=(e_1-e_2)\,(e_1-e_3)\,\frac{\sigma(b-c)\,\sigma(c-a)\,\sigma(a-b)}{\sigma_1(b-c)\,\sigma_1(c-a)\,\sigma_1(a-b)},\\ y_1&=\frac{\sigma(b-c)\,\sigma_1(u-b)\,\sigma_1(u-c)}{\sigma_1(b-c)\,\sigma(u-b)\,\sigma(u-c)},\\ &\text{and similar expressions for } y_2 \text{ and } y_3. \end{aligned}\right\} \tag{9}$$

If we substitute $u+\omega_1$ for u in equation (8), and for $\dfrac{\sigma_1(u-b)}{\sigma(u-b)}$ its corresponding value $\dfrac{\sigma(u-b)}{\sigma_1(u-b)}\sqrt{(e_1-e_2)\,(e_1-e_3)}$, and then divide the result by $(e_1-e_2)\,(e_1-e_3)$, we obtain the equation:

$$\frac{\sigma^2(b-c)\sigma^2(c-a)\,\sigma^2(a-b)}{\sigma_1^2(b-c)\sigma_1^2(c-a)\sigma_1^2(a-b)}\,(e_1-e_2)\,(e_1-e_3)\,\frac{1}{y_0}+\sum^{3}\frac{\sigma^2(b-c)}{\sigma_1^2(b-c)}\,\frac{1}{y_1}=0,$$

which is manifestly of the form $\left(\dfrac{A^2}{y}\right)=0$, where

$$\left.\begin{aligned} A_{00}&=(e_1-e_2)\,(e_1-e_3)\,A_{11}\,A_{22}\,A_{33},\\ A_{ii}&=\frac{\sigma^2(b-c)}{\sigma_i^2(b-c)},\quad i=1,2,3. \end{aligned}\right\} \tag{10}$$

By means of these equations (9) and (10) we shall now get an equation of that part of the double curve on Ω, which is the locus of the intersections of those generators of Ω which meet in perpendicular pairs.

§ 7. *The Double Curve.*

The plane section of Ω, as stated in § 4, has twenty-seven double points, and so the surface Ω must have a curve of double points of the 27th degree. At least a part of this double curve will be the locus of the vertices of the quadrics of this paper. This part we shall now find. We first seek the locus of the tangent planes of the quadrics at their vertices. Since the principal generators meet the plane W in points which, with the corresponding point y where the axis of the quadric pierces W, form a self-conjugate triangle, the plane on these two generators will meet W in a line that is the polar line of y. Consequently, the polar plane of y as to, say the circumsphere of T, will be a plane parallel to the principal tangent plane of the quadric above.

The equation of this circumsphere is given in quadri-planar coordinates as:

$$\frac{e_{21}^2\bar{x}_1\bar{x}_2}{x_1'x_2'}+\frac{e_{20}^2\bar{x}_2\bar{x}_0}{x_2'x_0'}+\ldots+\frac{e_{23}^2\bar{x}_2\bar{x}_3}{x_2'x_3'}=0,*$$

where e_{ij} is an edge of T, and x_i' is the altitude to the face $x_i=0$. In barycentric projective coordinates, this equation is:

$$e_{21}^2x_1x_2+\ldots+e_{23}^2x_2x_3=0, \text{ or } \overset{6}{\Sigma}e_{ij}^2x_ix_j=0,$$

where ($i=0, 1, 2, 3$, and $j=0, 1, 2, 3$).

The polar plane of y as to this sphere is:

$$\overset{6}{\Sigma}e_{ij}^2(x_iy_j+x_jy_i)=0,$$

which for convenience we shall write as $(\xi x)=0$ where

$$\xi_i=(e_{ij}^2y_j+e_{ik}^2y_k+e_{im}^2y_m) \qquad (j\neq k\neq m).$$

To get the planes tangent to the quadrics $\left(\frac{x^2}{y}\right)=0$ at their vertices, it is only necessary to get the plane parallel to $(\xi x)=0$, which touches the quadric. Any plane parallel to $(\xi x)=0$, is given by

$$(\xi x)+\lambda(x)=0. \tag{11}$$

The condition for this plane to touch the quadric $\left(\frac{x^2}{y}\right)=0$ reduces at once to

$$\lambda^2(y)+2\lambda(\xi y)+(\xi^2 y)=0, \tag{12}$$

where the y's are subjected to the two relations $(y)=0$ and $\left(\frac{A^2}{y}\right)=0$.

Substituting the values for ξ_i in $(\xi^2 y)$ this term reduces to

$$\overset{4}{\Sigma}y_1y_2y_3[2(e_{12}^2e_{23}^2+e_{23}^2e_{31}^2+e_{31}^2e_{12}^2)-e_{12}^4-e_{13}^4-e_{23}^4].$$

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* "Rogers's revision of Salmon's Geometry of Three Dimensions," p. 235.

As the coefficient of $y_1y_2y_3$ is sixteen times the square of the area of the face $x_0=0$ of T, *i. e.*, equals $16A_{00}$, this last quantity can be written as

$$16y_0y_iy_2y_3\left(\frac{A^2}{y}\right).$$

Therefore $(\xi^2y)=16y_0y_1y_2y_3\left(\frac{A^2}{y}\right)=0$, and (12) reduces to $2\lambda(\xi y)=0$. This last equation has the roots $\lambda=0, \infty$. If $\lambda=\infty$, (11) reduces to $(x)=0$, the plane W already discussed. If $\lambda=0$, (11) becomes $(\xi x)=0$, the principal tangent plane of $\left(\frac{x^2}{y}\right)=0$, and as y traces out the cubic curve $\left(\frac{A^2}{y}\right)=0$ in the plane W, $(\xi x)=0$ gives the locus of these planes.

The y's, as given by equations (9), are cubics in σ, and as $(\xi x)=0$ is linear in y, when the σ's are substituted the equation resulting will be a cubic in σ. Therefore *these double tangent planes of Ω envelope a cubic curve.*

An interesting property of these double tangent planes is derived very simply. The points of any plane cubic reciprocate, with regard to a sphere, into planes on a cone of class 3, with its vertex at the point which is the reciprocal of the plane on which the cubic lies. In this case the latter plane is the plane W which reciprocates with regard to the sphere circumscribing T into the centre of this sphere. Thus:

All the planes $(\xi x)=0$ pass through the centre of the sphere circumscribing the reference tetrahedron T.

The equation $\left(\frac{x^2}{y}\right)=0$, in planes, is $(\xi^2y)=0$. The pole of ξ, one of the planes enveloping $(\xi x)=0$, will be the vertex of the quadric of which ξ is the principal tangent plane. Taking the coefficients of ξ' in $(\xi\xi'y)=0$ as the coordinates of x, we have $x_i=\xi_iy_i$ as the locus of the vertices of all the quadrics $\left(\frac{x^2}{y}\right)=0$. This is a quadratic in y and therefore a sextic in σ, *i. e.*:

The locus of the intersections of those generators of Ω which meet in perpendicular pairs, is an elliptic space sextic.

The six points where this sextic meets the plane W are the points where the cubic R meets the absolute. It is at once seen that the tangent to R at any one of these points—which is a line of Ω, as before stated—is perpendicular to that generator of Ω which pierces W at that point.

As the mid-points of the edges of T are points on this sextic, we have at once that three of the points where this sextic pierces any face of T, are the mid-points of the edges of T which lie in that face.

The complete curve of double points being of degree 27, there remains a part of degree $27-6=21$, which is the intersection of those generators which are not perpendicular. This part meets each generator of Ω seven times. Also since the curve of double planes is of degree 3 less than the degree of the curve of double points, the complete curve of double planes would be of degree 24. Of this the cubic $(\xi x)=0$, has already been accounted for, there remaining a part of degree 21.

§ 8. *A Quadratic Transformation Connected with* Ω.

The equation of the sextic curve of double points suggests a quadratic transformation. Let this equation be written as

$$x_i=\alpha_{ij}y_iy_j+\alpha_{ik}y_iy_k+\alpha_{il}y_iy_l, \tag{13}$$

where $\alpha_{ij}=e_{ij}^2$ and $i, j, k, l=0, 1, 2, 3$, $i\neq j\neq k\neq l$. If in $(\xi x)=0$, the relation of incidence between a point x and a plane ξ, we substitute these values for x_i, the correspondence established by (13) takes the form

$$\sum^{3}(\alpha_{ij}y_iy_j\xi_i+\alpha_{ik}y_iy_k\xi_i+\alpha_{il}y_iy_l\xi_i)=0. \tag{14}$$

Let us regard this as a correspondence between two spaces s_x and s_y. To a plane ξ in s_x corresponds a quadric in s_y. To a line in s_x corresponds a quartic curve in s_y. To a point in s_x corresponds four points in s_y. A plane in s_y meeting a quartic curve in four points, the correspondent in s_x must meet a line in four points; and, therefore, must be a quartic surface which can be shown to be a Steiner's Quartic Surface. To a line in s_y corresponds a conic in s_x. A plane η meets the three-fold system of quadrics in a three-fold system of conics, which map the plane η onto the corresponding Steiner's Quartic Surface.

The Jacobian of the system of quadrics is a quartic surface whose equation is

$$J\equiv\begin{vmatrix} \alpha_{01}y_1+\alpha_{02}y_2+\alpha_{03}y_3, & \alpha_{01}y_0, & \alpha_{02}y_0, & \alpha_{03}y_0, \\ \alpha_{01}y_1, & \alpha_{01}y_0+\alpha_{12}y_2+\alpha_{13}y_3, & \alpha_{12}y_1, & \alpha_{13}y_1, \\ \alpha_{02}y_2, & \alpha_{12}y_2, & \alpha_{02}y_0+\alpha_{12}y_1+\alpha_{23}y_3, & \alpha_{23}y_2, \\ \alpha_{03}y_3, & \alpha_{13}y_3, & \alpha_{23}y_3, & \alpha_{03}y_0+\alpha_{13}y_1+\alpha_{23}y_2, \end{vmatrix}=0.$$

Or,

$$\begin{aligned} J\equiv{}&\alpha_{01}\alpha_{02}\alpha_{12}y_0y_1y_2(\alpha_{03}y_0+\alpha_{13}y_1+\alpha_{23}y_2)\\ &+\alpha_{01}\alpha_{03}\alpha_{13}y_0y_1y_3(\alpha_{02}y_0+\alpha_{12}y_1+\alpha_{23}y_3)\\ &+\alpha_{02}\alpha_{03}\alpha_{23}y_0y_2y_3(\alpha_{01}y_0+\alpha_{12}y_2+\alpha_{13}y_3)\\ &+\alpha_{12}\alpha_{13}\alpha_{23}y_1y_2y_3(\alpha_{01}y_0+\alpha_{02}y_2+\alpha_{03}y_3)=0. \end{aligned} \tag{15}$$

Or,

$$J\equiv\sum^{4}\alpha_{ij}\alpha_{ik}\alpha_{jk}y_iy_jy_k(\alpha_{il}y_i+\alpha_{jl}y_j+\alpha_{kl}y_k)=0.$$

Since to a line in s_x corresponds a quartic in s_y, the correspondent of J will meet the line in the same number of points as the number in which the quartic curve will meet J, or in sixteen points. Then the correspondent of J is a surface of order 16. A plane ξ will touch this latter surface whenever J touches the quadrics of the system, *i. e.*, when the discriminant of (15) vanishes. The equation of the correspondent of J is given in planes, by bordering this discriminant, and is

$$\sum^{3}\alpha_{01}^2\alpha_{23}^2(\xi_0+\xi_1)^2(\xi_2+\xi_3)^2-2\sum^{3}\alpha_{01}\alpha_{02}\alpha_{13}\alpha_{23}(\xi_0+\xi_1)(\xi_0+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)=0,$$

or, which can be written

$$Q\equiv\sqrt{\alpha_{01}\alpha_{23}(\xi_0+\xi_1)(\xi_2+\xi_3)}+\sqrt{\alpha_{02}\alpha_{13}(\xi_0+\xi_2)(\xi_1+\xi_3)} \\ +\sqrt{\alpha_{03}\alpha_{12}(\xi_0+\xi_3)(\xi_1+\xi_2)}=0. \qquad (16)$$

As this is of the fourth degree in ξ the correspondent of the Jacobian is a surface of class 4. Since the reflection of the plane $\xi_0+\xi_1=0$ in the centroid of T, whose coordinates are (1, 1, 1, 1), is the plane $\xi_2+\xi_3$, and similarly for $\xi_0+\xi_2$ and $\xi_0+\xi_3$, which reflect into the planes $\xi_1+\xi_3$ and $\xi_1+\xi_2$, Q reflects into itself, and is therefore symmetrical about this centroid. Thus:

The correspondent of the Jacobian of the system of quadrics is a surface of order 16 *and class* 4, *which is symmetrical with regard to the centroid of T.*

The equation of the Steiner's Quartic Surface corresponding to the plane W is found by forming the Hessian of (14), changing the resulting equations in planes into the point equation, and making this surface touch W_x. The result is

$$R\equiv(a_{12}+a_{13}-a_{23})(a_{01}a_{23}+a_{02}a_{03})+(a_{12}-a_{13}+a_{23})(a_{02}a_{13}+a_{01}a_{03}) \\ +(-a_{12}+a_{13}+a_{23})(a_{03}a_{12}+a_{01}a_{02})=a_{01}^2a_{23}+a_{02}^2a_{13}+a_{03}^2a_{12}+a_{12}a_{13}a_{23}, \qquad (17)$$

where

$$a_{ij}=\alpha_{ij}(\xi_i+\xi_j).$$

The cubic c_3, *i. e.*, the intersection of W and Ω, maps into the sextic curve $x_i=\xi_i y_i$, the curve of double points of § 7. The quadric corresponding to the plane W, whose coordinates are (1, 1, 1, 1), is seen from (14) to be $(\xi y)=0$, *i. e.*, is the circumsphere of T. This is one of the quadrics of the system and consequently the absolute is one of the conics of the system which map the plane W onto its corresponding surface R. Since to the plane W in s_y corresponds the quartic surface R in s_x and to W in s_x corresponds the quadric $(\xi y)=0$ in s_y, to the intersection of W and (ξy), *i. e.*, to the absolute, will correspond the intersection of W and R, *i. e.*, a rational quartic on R. As the

absolute meets the cubic c_3 in six points, this quartic on R will be on six points of the sextic $x_i = \xi_i y_i$. These two sets of six points are one and the same.

To the four lines $y_i = 0$, which are the lines of W cut out by the planes of T, correspond a set of four lines on R whose equations are

$$e_{01}^2 y_1 + e_{02}^2 y_2 + e_{03}^2 y_3 = 0,$$

and three similar ones.

§ 9. *Surfaces Connected with* Ω.

If three points are chosen, whose Cartesian coordinates are (a, b, c), (d, e, f), (r, s, t) and their polar planes, with regard to the quadric $xy = z$, are determined, they will meet in a fourth point, whose coordinates are $\left(\frac{B}{C}, -\frac{A}{C}, -\frac{D}{C}\right)$, where A, B, C, D are the first minors of $x, y, z, 1$, respectively in

$$\begin{vmatrix} x & y & z & 1 \\ a & b & c & 1 \\ d & e & f & 1 \\ r & s & t & 1 \end{vmatrix} = 0.$$

If we put the z-coordinates of these four points, which we shall consider as the vertices of a tetrahedron T, equal to zero, the resulting coordinates represent the projections of the points in the plane, $z = 0$. This is the principal tangent plane of the above quadric. These four projected points lie on a circle, for if the coordinates are substituted in the determinant

$$\Delta \equiv \begin{vmatrix} a^2 + b^2, & a & b & 1 \\ d^2 + e^2, & d & e & 1 \\ r^2 + s^2, & r & s & 1 \\ \left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 & \frac{B}{C}, & -\frac{A}{C} & 1 \end{vmatrix},$$

which vanishes if the points are on a circle, Δ becomes

$$\Delta \equiv (a^2+b^2)(e^2r^2 - d^2s^2) + (d^2+e^2)(a^2s^2 - b^2r^2) + (r^2+s^2)(b^2d^2 - a^2e^2) \equiv 0.$$

$\therefore$ *The vertices of* T *are on a circular cylinder which is perpendicular to the principal tangent planes of all the quadrics* $\left(\frac{x^2}{y}\right) = 0$.

The reflections of the vertices of T in either of the principal generators of $xy = z$; *i. e.*, in either of the lines $x = 0$, or $y = 0$, will give a new tetrahedron which is inscribed to T, *i. e.*, of the kind called T' in the first part of this paper. Let them be T' and T''. It is easily seen, either analytically or geometrically,

that the vertices of T' and T'' are also on circular cylinders perpendicular to the plane $z=0$. It follows directly from the way in which T' and T'' are determined that T, T' and T'' are symmetrically placed with regard to the z-axis of the quadric.

Thus, T, T' *and* T'' *are inscribed in circular cylinders of equal radii, all of which are perpendicular to the principal tangent plane of the quadric* $xy=z$, *and the axes of these three cylinders are on a circular cylinder whose axis is the* z*-axis of the quadric.*

For a point to lie on a cylinder is one condition, and as five conditions determine a cylinder, for it to be on four points leaves one degree of freedom. Then there are ∞^1 cylinders on four points. Beltrami has shown that axes of such cylinders meet the plane at infinity in the cubic curve $\left(\frac{A^2}{y}\right)=0$. The axes of the above cylinders will all meet W on this same cubic.

The axes of the circular cylinders on T form a ruled surface of the ninth order.* Each generator of this surface is then perpendicular to a double plane of Ω, and conversely, each double plane of Ω is perpendicular to one generator of the above surface.

The axes of the paraboloids $\left(\frac{x^2}{y}\right)=0$ also form a ruled surface whose order *can not be more than nine*, for its generators pass through and are determined by the sextic curve of double points, and the cubic curve $\left(\frac{A^2}{y}\right)=0$ at infinity.

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* J. B. Eck, "Über die Verteilung der Axen der Rotationsflächen zweiten Grades, welche durch gegebene Punkte gehen" (Dissertation).